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## **The Dissipation Range in Rotating Turbulence**

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# THE DISSIPATION RANGE IN ROTATING TURBULENCE\*

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**Abstract.** The dissipation range energy balance of the direct interaction approximation is applied to rotating turbulence when rotation effects persist well into the dissipation range. Assuming that  $RoRe^{1/2} \ll 1$  and that three-wave interactions are dominant, the dissipation range is found to be concentrated in the wavevector plane perpendicular to the rotation axis. This conclusion is consistent with previous analyses of inertial range energy transfer in rotating turbulence, which predict the accumulation of energy in those scales.

**Key words.** rotating turbulence, dissipation range

**Subject classification.** Physical Sciences

**1. Introduction.** Kraichnan [1] demonstrated that despite the analytical complexity of the equations of the direct interaction approximation, solutions representing the universal inertial and dissipation ranges of turbulence are readily accessible. In the inertial range, where energy transfer vanishes, the closure equations are satisfied by a power law spectrum with a constant energy flux from large to small scales of motion.

In the dissipation range, viscous dissipation balances the energy input to each scale from nonlinear interaction. By assuming that the turbulent time scale is the viscous time scale  $\Theta \sim (\nu k^2)^{-1}$ , the dissipation-range spectrum

$$(1.1) \quad E(k) \sim k^3 \exp(-\beta k/k_d)$$

is obtained. In Eq. (1.1),  $k_d = (\varepsilon/\nu^3)^{1/4}$  is the Kolmogorov scale;  $\beta$  will be used throughout to denote a universal constant, but not the same constant each time it appears. This spectrum is consistent with experimental and numerical data [2, 3]. The theory also predicts that nonlinear interactions in the dissipation range are predominantly among nearly collinear wavevector triads; this conclusion is evaluated in [4, 5, 6].

Kraichnan [1] also proposed a near-dissipation range balance in which a nonlinear time scale replaces the viscous time scale. If this time scale is determined, consistently with Eulerian DIA by the sweeping hypothesis  $\Theta \sim (V_0 k)^{-1}$ , the energy spectrum  $E(k) \sim k^2 \exp(-k/k_d)$  is obtained. If the Kolmogorov time-scale is used instead, the result would be

$$(1.2) \quad E(k) \sim k^{5/3} \exp(-k/k_d)$$

Sirovich *et al.* [7] propose an energy spectrum covering both the inertial and dissipation ranges which reduces to Eq. (1.2) in the dissipation range.

**2. Analysis.** In rotating turbulence, the additional time-scale introduced by the Coriolis force makes a variety of dissipation range balances possible. Thus, although the very smallest scales will always be subject

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to the viscous time-scale, leading to the spectrum of Eq. (1.1), a different result might be anticipated when wave interactions dominate nonlinearity well into the dissipation range.

The standard elementary scaling arguments suggest when this condition applies. The wave frequency in rotating turbulence is determined by the dispersion relation of inertial waves as  $\omega(\mathbf{k}) = \Omega k_z/k$ . To begin, we will ignore the angular dependence of this frequency. Assuming the spectrum [8] for rapid rotation  $E(k) \sim (\varepsilon\Omega)^{1/2}k^{-2}$  and defining the rotational dissipation scale  $k_{d,\Omega}$  by the condition

$$(2.1) \quad \int_0^{k_{d,\Omega}} \nu k^2 E(k) dk = \varepsilon$$

there results  $k_{d,\Omega} \sim (\varepsilon/\Omega\nu^2)^{1/2}$ . The viscous frequency scale exceeds the wave frequency scale at  $k'_{d,\Omega}$  defined by  $\nu(k'_{d,\Omega})^2 \sim \Omega$ . Assume that this scale is much larger than  $k_{d,\Omega}$  so that

$$(2.2) \quad k_{d,\Omega}/k'_{d,\Omega} \sim \frac{\varepsilon^{1/2}}{\Omega\nu^{1/2}} \sim RoRe^{1/2} \ll 1$$

and consider the scales satisfying  $k_{d,\Omega} < k < k'_{d,\Omega}$ .

The direct interaction approximation (DIA) predicts the dissipation range energy balance [1]

$$(2.3) \quad 2\nu k^2 Q(\mathbf{k}) = \frac{1}{4} \int d\mathbf{p} d\mathbf{q} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \int_0^\infty d\tau \times \\ P_{imn}(\mathbf{k}) P_{i'rs}(\mathbf{k}) Q_{mr}(\mathbf{p}, \tau) Q_{ns}(\mathbf{q}, \tau) G_{ii'}(\mathbf{k}, \tau)$$

Standard notation is used in Eq. (2.3):  $Q_{ij}(\mathbf{k}, \tau) = \langle u_i(\mathbf{k}, t) u_j(\mathbf{k}', s) \rangle / \delta(\mathbf{k} + \mathbf{k}')$  is the two-time correlation function, where  $\tau$  denotes the time difference  $\tau = t - s$ . The single-time correlation function is denoted by  $Q_{ij}(\mathbf{k})$ , and the DIA response function is  $G_{ij}(\mathbf{k}, \tau)$ . The tensorially isotropic form  $Q_{ij}(\mathbf{k}) = Q(\mathbf{k}) P_{ij}(\mathbf{k})$  is assumed, where  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$  and finally,  $P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$ .

Eq. (2.3) is simply the general DIA energy balance, with eddy damping ignored in comparison to viscous damping. The assumption Eq. (2.2) means that wave turbulence theory [9] applies in the dissipation range. Therefore, the DIA response function  $G_{ij}$  is simply the linear response

$$(2.4) \quad G_{ij}(\mathbf{k}, \tau) = \{\cos(2\Omega k_z \tau)/k \xi_{ij}^1(\mathbf{k}) + \sin(2\Omega k_z \tau)/k \xi_{ij}^0(\mathbf{k})\} H(\tau)$$

where  $H$  is the unit step function and the tensors  $\xi^i$  are defined in terms of the *Craya-Herring* basis

$$(2.5) \quad \mathbf{e}^{(1)}(\mathbf{k}) = \mathbf{k} \times \boldsymbol{\Omega} / |\mathbf{k} \times \boldsymbol{\Omega}| \\ \mathbf{e}^{(2)}(\mathbf{k}) = \mathbf{k} \times (\mathbf{k} \times \boldsymbol{\Omega}) / |\mathbf{k} \times (\mathbf{k} \times \boldsymbol{\Omega})|$$

by

$$(2.6) \quad \xi_{ij}^0 = e_i^{(1)} e_j^{(2)} - e_j^{(1)} e_i^{(2)} \\ \xi_{ij}^1 = e_i^{(1)} e_j^{(1)} + e_i^{(2)} e_j^{(2)}$$

Note that  $\xi_{ij}^1 = P_{ij}(\mathbf{k})$ . At lowest order in the weak turbulence approximation,

$$(2.7) \quad Q_{ij}(\mathbf{k}, \tau) = G_{im}(\mathbf{k}, \tau) Q_{mj}(\mathbf{k}) + G_{jm}(\mathbf{k}, -\tau) Q_{mi}(\mathbf{k})$$

For the correlation function  $Q(\mathbf{k})$ , we provisionally adopt Kraichnan's dissipation range hypothesis [1]

$$(2.8) \quad Q(k) = \kappa (k/k_{d,\Omega})^\alpha e^{-(\beta k/k_{d,\Omega})}$$

where the dimensional constant  $\kappa$  will be determined by the calculation. Substituting Eqs. (2.4)-(2.8) in the dissipation range balance Eq. (2.3), the right side will contain time integrals of the form

$$(2.9) \quad R(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \int_{-\infty}^{\infty} d\tau \exp i\tau\Omega \left\{ \frac{k_z}{k} \pm \frac{p_z}{p} \pm \frac{q_z}{q} \right\}$$

where the lower limit of integration insures that the result is real. We can assume [10]

$$(2.10) \quad R(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \frac{1}{\Omega} \delta\left(\frac{k_z}{k} \pm \frac{p_z}{p} \pm \frac{q_z}{q}\right)$$

As Eq. (2.10) shows, the integral in Eq. (2.3) vanishes unless the triad  $\mathbf{k}, \mathbf{p}, \mathbf{q}$  satisfies the *resonance condition*

$$(2.11) \quad \frac{k_z}{k} \pm \frac{p_z}{p} \pm \frac{q_z}{q} = 0$$

But as Kraichnan [1] notes, the exponential *ansatz* Eq. (2.8) forces the triad to be nearly collinear: the quantities  $\exp(-\beta k)$  on the left side of Eq. (2.3) and  $\exp(-\beta(|\mathbf{p}| + |\mathbf{q}|))$  on the right can be comparable only if  $|\mathbf{k}| \approx |\mathbf{p}| + |\mathbf{q}|$ , whence the triangle inequality implies near collinearity of the vectors  $\mathbf{k}, \mathbf{p}, \mathbf{q}$ . In this case,

$$(2.12) \quad \begin{aligned} \mathbf{p} &= \pm \frac{p}{k} \mathbf{k} + \mathbf{r} \\ \mathbf{q} &= \pm \frac{q}{k} \mathbf{k} - \mathbf{r} \end{aligned}$$

where  $\pm p \pm q = k$ , and  $\mathbf{p} \cdot \mathbf{r} = 0$ , therefore

$$(2.13) \quad \begin{aligned} |\mathbf{p}| &\approx p\left(1 + \frac{r^2}{p^2}\right) \\ |\mathbf{q}| &\approx q\left(1 + \frac{r^2}{q^2}\right) \end{aligned}$$

In view of Eq. (2.8), the integral in Eq. (2.3) decays exponentially with  $r$ .

It is immediately evident that substitution of Eq. (2.12) into the resonance condition Eq. (2.11) yields the contradiction  $\pm 1 \pm 1 \pm 1 = 0$  unless  $k_z$  is nearly zero. Thus, the resonance and collinearity conditions can only be satisfied by nearly horizontal wavevector triads. The right side of Eq. (2.3) is therefore nearly zero for non-horizontal wavevectors  $\mathbf{k}$ , contradicting the assumption that the left side is isotropic.

But by assuming instead that the energy spectrum is planar, so that instead of Eq. (2.8),

$$(2.14) \quad Q(\mathbf{k}) = \kappa(k/k_{d,\Omega})^\alpha e^{-(\beta k/k_{d,\Omega})} \delta(k_z)$$

we find that Eq. (2.3) can be satisfied, since

$$(2.15) \quad \int d\mathbf{p} d\mathbf{q} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \delta(p_z) \delta(q_z) = \int_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d\mathbf{p} \delta(p_z) \delta(k_z)$$

If the dissipation range excitation is confined to the plane  $k_z = 0$ , the wave time-scale does not apply since the planar modes are unaffected by rotation. The dissipation range balance must be computed assuming the viscous time-scale instead. With this understanding, the integrations in Eq. (2.3) are easily performed using the variables  $p, \mathbf{r}$  in place of  $\mathbf{p}$ , and lead to the result

$$(2.16) \quad \kappa k^{2\alpha+1} k_{d,\Omega}^{1-2\alpha} / \nu \sim \nu k^2 (k/k_{d,\Omega})^\alpha$$

where the common exponential factor has been cancelled from both sides and a common factor of  $\delta(k_z)$  is cancelled using Eq. (2.15). The powers of  $k$  balance if  $\alpha = 1$ , consequently  $\kappa = \nu^2$ . Therefore, the correlation function is

$$(2.17) \quad Q(\mathbf{k}) = C\nu^2(k/k_{d,\Omega})e^{-\beta(k/k_{d,\Omega})}\delta(k_z)$$

The effect of rotation is indirect: rotation excludes certain classes of vectors from the dissipation range, but the dissipation range dynamics in the remaining scales is independent of rotation.

This conclusion was based on the impossibility of satisfying the three-wave resonance conditions by nearly collinear vectors, except when those vectors are nearly horizontal. In problems in which the dispersion relations do not permit three-wave interactions, perturbation theory leads to a modified equation of motion for waves with a higher order nonlinearity than the governing equations [11]. If the governing equations are quadratically nonlinear, the result is a theory with a cubic nonlinearity. For such a theory of four-wave interactions, nontrivial resonances are always possible.

It is therefore natural to ask whether four-wave interactions could play a role in the dissipation range of rotating turbulence. The possible role of four-wave interactions in the inertial range of rotating turbulence has been suggested by Yakhot [12]. The dissipation range balance for four-wave interactions, which replaces Eq. (2.3) has the form

$$(2.18) \quad \begin{aligned} 2\nu k^2 Q_{ij}(\mathbf{k}) = & \int d\mathbf{p} d\mathbf{p}' d\mathbf{q} \delta(\mathbf{k} - \mathbf{p} - \mathbf{p}' - \mathbf{q}) \int_0^\infty dt \int_0^\infty dt' \int_0^\infty ds \times \\ & P_{imn}(\mathbf{k}) P_{m'rs}(\mathbf{p}) P_{j'kl}(\mathbf{k}) P_{k'pq}(\mathbf{p}) G_{mm'}(\mathbf{p}, t - s) G_{kk'}(\mathbf{p}, t' - s') \times \\ & G_{jj'}(\mathbf{k}, t - t') Q_{rp}(\mathbf{p}', s - s') Q_{sq}(\mathbf{p} - \mathbf{p}', s - t) Q_{nl}(\mathbf{q}, t - s) \end{aligned}$$

in which only one representative term is written on the right side. The assumption of near collinearity takes the form

$$(2.19) \quad \begin{aligned} \mathbf{p} &= \pm \frac{p}{k} \mathbf{k} + \mathbf{r} \\ \mathbf{p}' &= \pm \frac{p'}{k} \mathbf{k} + \mathbf{r}' \\ \mathbf{q} &= \pm \frac{q}{k} \mathbf{k} - \mathbf{r} \end{aligned}$$

Quartets of this form can always be found to satisfy the four-wave resonance condition

$$(2.20) \quad \frac{k_z}{k} \pm \frac{p_z}{p} \pm \frac{p'_z}{p'} \pm \frac{q_z}{q} = 0$$

Substituting the exponential *ansatz* Eq. (2.8) in the four-wave dissipation range balance Eq. (2.18),

$$(2.21) \quad k^6 k_{d,\Omega}^4 (k/k_{d,\Omega})^{3\alpha} \Omega^{-3} \kappa^2 = \nu k^2 (k/k_{d,\Omega})^\alpha$$

Balancing the powers of  $k$  gives  $\alpha = -2$ . The definition of  $k_{d,\Omega}$  implies

$$(2.22) \quad Q(k) = C'_D \frac{\Omega^{7/2} \nu^{9/2}}{\varepsilon^2} (k/k_{d,\Omega})^{-2} e^{-\beta' k/k_{d,\Omega}}$$

In problems in which three- and four-wave interactions are both present, it can be assumed that the three-wave processes are dominant [13]. In this case, the conclusion that the dissipation range spectrum contains only horizontal vectors is consistent with the picture of energy transfer in rotating turbulence

proposed on the basis of closure studies by Cambon *et al.* [14, 15], namely that in the inertial range, energy is transferred to the wavenumber plane perpendicular to the rotation axis. Further theoretical support for this conclusion was provided by the *instability principle* of Waleffe [9].

The impossibility of dissipating energy in nearly vertical directions by energy transfer to smaller scales would naturally force the transfer of energy toward the horizontal plane, where viscous dissipation is possible. Although this picture of energy transfer in the inertial and dissipation ranges is self-consistent, it must be stressed that the conclusions about inertial range transfer do not require the strong restriction on the Rossby number required here.

It should be noted that the prediction that the dissipation range is restricted to exactly horizontal vectors is a consequence of ignoring the angular dependence of the wave time-scale. In fact, since the wave frequency is  $\Omega k_z/k$ , the condition  $k_{d,\Omega} \geq k_d$  occurs for wavevectors satisfying  $k_z/k \leq \cos \theta$  where  $\cos \theta = \varepsilon^{1/2}/\nu^{1/2}\Omega$ . For these modes, the viscous time-scale is dominant, and the three-wave dissipation range extends to the region  $-k \cos \theta \leq k_z \leq k \cos \theta$  instead of to the plane  $k_z = 0$ . Note also that this argument shows that the scale  $k_{d,\Omega}$  in Eq. (2.17) should be replaced by the Kolmogorov scale  $k_d$ .

The approximate two-dimensionalization of the small scales of rotating turbulence under the limit defined by Eq. (2.2) raises the question of the relationship between the present results and the Taylor-Proudman theorem [16]. As noted by Smith and Waleffe [17], the large scales of rapidly rotating turbulence are always subject to the Taylor-Proudman theorem since they are nearly steady. Whereas the Taylor-Proudman theorem requires the applicability of steady, linear dynamics, the present argument based on wave interactions allows both unsteadiness and nonlinearity. The possibility that a combination of large-scale two-dimensionalization due to the Taylor-Proudman theorem with two-dimensionalization of the small scales due to the impossibility of certain three-wave interactions leads to two-dimensionalization of all scales of motion in the extreme limit  $RoRe^{1/2} \ll 1$  is an interesting theoretical possibility which warrants further investigation.

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